Gabor Szekelyhidi, The Calabi Functional on a Ruled Surface 2007

Background: Trying to understand geometry of Kähler mflds
When does a mfld possess special/canonical metrics with nice curvature properties?

(X, ω) cpt Kähler, $\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$

Ricci curv. $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det g$, Scalar curv. $R = g^{i\bar{j}} R_{i\bar{j}}$

$[Ric] = c_1(X)$

~~Does~~ Can ask: in $[\omega]$, does there exist a metric of constant scalar curv?
4th order nonlinear eqn, ~~how~~ very hard

More known about metrics in canonical class
If $c_1(X) = c$ pos, neg or 0, look for KE metrics st $R_{i\bar{j}} = c g_{i\bar{j}}$ (continuity method, KRF)

More gen: Calabi functional: $Ca(g) = \int_X R^2 \omega^n$
modified Calabi functional $\tilde{Ca}(g) = \int_X (R - \hat{R})^2 \omega^n$
Look for min/extremizers of this functional

$\delta R = -L(\delta\varphi) + \langle \nabla R, \nabla \delta\varphi \rangle$

KE $\subset$ csc $\subset$ extremal

Calabi flow ~~is gradient flow of Calabi functional~~

$\dot{g}_{i\bar{j}} = \partial_i \partial_{\bar{j}} R$

If $g_{i\bar{j}} = (g_0)_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$, $\dot{\varphi} = R - \hat{R}$, $\frac{d}{dt} Ca(g_t) = -\int (LR, R) \omega_t^n$

L is Lichnerowicz operator, $L = \Delta^2 + Ric^{i\bar{j}} \partial_i \partial_{\bar{j}} + \langle \nabla R, \nabla \cdot \rangle$

Try to ~~find~~ compute examples
On a ruled surface, can exploit symmetry to simplify problem

A ruled surface is a $\mathbb{P}^1$-bundle over a cx curve, an example of a good mfld for use of the Calabi ansatz.

General construction: (see Hwang-Singer)

$\tilde{X}$ = total space of hermitian hol line bundle over Kähler mfld

data $p : (\overset{w}{L}, h) \to (\overset{z}{M}, \omega_M)$

Calabi ansatz: look at metrics of the form $\omega = p^*\omega_M + 2i\partial\bar\partial f(s)$

where $s = \frac{1}{2}\log|(z,w)|^2 = \frac{1}{2}\log|w|^2 + \frac{1}{2}\log h(z)$

Such a metric is invariant under the natural $U(1)$ action on the fibers

Illustrate the use of this construction in Gabor's example, general vs. specific features will be clear or pointed out.

Let Σ be a Riemann surface of genus 2 w/ (Kähler) metric ω,

$\int_\Sigma \omega_\Sigma = 2\pi$, or $[\frac{\omega}{2\pi}] = 1 \in H^2(\Sigma, \mathbb{Z})$

ω has csc -2 : $\int R\,\omega_\Sigma = 2\pi(2-2g) = -4\pi$

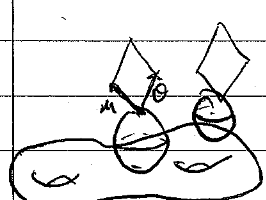
$(M, h) \to (\Sigma, \omega)$ a line bundle of deg. -1, herm. metric h

Curv. form of $\overset{\Theta}{h} := -\partial\bar\partial \log h = i\omega_\Sigma$

$c_1(\Sigma) = -2$

Note: $c_1(M) = [\frac{i}{2\pi}\Theta] = [\frac{-\omega}{2\pi}] = -1$

Our ruled surface will be $X = \mathbb{P}(M \oplus \mathcal{O}) \to (\Sigma, \omega_\Sigma)$



Divisors: Fiber C, infinity section $S_\infty = [1,0]$

$\omega = p^{\gamma}\omega_{\Sigma} + 2i\partial\bar\partial f(s)$, $\quad s = \frac{1}{2}\log|w|^2 + \frac{1}{2}\log h(z)$

Two equiv. ways of describing the construction of a <u>momentum profile</u>:

1. <u>Via moment map</u>

s, and therefore f & ω, are invariant under rotation ($U(1)$-action) on fiber,
gen by v.f. $X = \frac{\partial}{\partial\theta_w} = i\left(w\frac{\partial}{\partial w} - \bar w\frac{\partial}{\partial \bar w}\right)$

$\Rightarrow \mathcal{L}_X\omega = 0$ $\quad\Rightarrow \iota_X\omega$ an exact 1-form

$d\iota_X\omega + \iota_X d\omega = d\iota_X\omega = 0 \Rightarrow \iota_X\omega = -d\tau$ for some function τ, const. on level sets of s

τ called the <u>moment map</u>.

Let $\varphi(\tau) := \|X\|^2_\omega$:

~~Can check by~~ Can choose coords st at a pt (z_0, w_0), $d\log h(z) = 0$

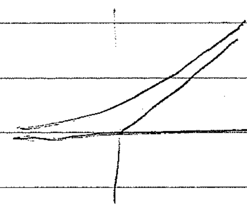
Then $2i\partial\bar\partial f(s) = if'(s)\partial\bar\partial\log h(z) + f''(s)\frac{i\,dw\wedge d\bar w}{2|w|^2}$

$\Rightarrow \varphi(\tau) = \omega(X, X) = f''(s)$

2. <u>Via Legendre transform</u>

f convex (needed to make $\overset{\text{\& } g}{\omega}$ pos-def) $\rightarrow$ can def Leg. trans

$F(\tau) = \hat f(\tau) := \sup_s(s\tau - f(s))$



f convex $\Rightarrow f'$ is increasing, or injective

Leg. trans. gives convex fn

$F'(\tau) = s$ where $\tau = f'(s)$

or $f(s) + F(\tau) = s\tau$

$\therefore F'(\tau) = s,\ F''(\tau) = \frac{\partial s}{\partial \tau}$

Let $\underline{\varphi(\tau)} := \frac{1}{F''(\tau)} = \frac{1}{\partial s/\partial\tau}$ $\quad$ Then take $\frac{\partial^2}{\partial s^2}\left(f(s) + F(\tau) = s\tau\right)$, get $f''(s) = \varphi(\tau)$

Importance of momentum profile:

rewrite · $2i\partial\bar\partial f(s) = if'(s)\partial\bar\partial \log h(z) + f''(s)\frac{i\,dw\wedge d\bar w}{2|w|^2}$

$$= \tau p^*\omega_\Sigma + \varphi(\tau)\frac{i\,dw\wedge d\bar w}{2|w|^2}$$

$$\Rightarrow \omega = (1+\tau)p^*\omega_\Sigma + \varphi(\tau)\frac{i\,dw\wedge d\bar w}{2|w|^2}$$

$$\log\frac{1}{|w|^2} = -2\cdot\left(\tfrac12\log|w|^2\right) = -2\left(s - \tfrac12\log h\right) = -2s+\log h$$

$$s = \tfrac12\log|w|^2 + \tfrac12\log h(z) \qquad \frac{\partial s}{\partial w} = \frac12\frac{\bar w}{|w|^2} = \frac{1}{2w}$$

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial s}\frac{\partial s}{\partial w} = \frac{1}{2w}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial \bar w} = \frac{1}{2\bar w}\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial s} = \frac{\partial\tau}{\partial s}\frac{\partial}{\partial\tau} = \varphi(\tau)\frac{\partial}{\partial\tau}$$

$$\frac{\partial}{\partial z} = \frac{\partial\tau}{\partial z}\frac{\partial}{\partial\tau} = \frac{\partial\tau}{\partial s}\frac{\partial s}{\partial z}\frac{\partial}{\partial\tau} = \varphi(\tau)\frac{\partial_z h}{2h}\frac{\partial}{\partial\tau}$$

· $\det(g) = \frac{1}{|w|^2}(1+\tau)\varphi(\tau)\det(g_\Sigma)$ valid at all pts

So Ricci form $\rho = -i\partial\bar\partial\log\det g$

$$= -i\partial\bar\partial\Big(\log\det g_\Sigma + \underbrace{\log\tfrac{1}{|w|^2}(1+\tau)\varphi(\tau)}_{\log((1+\tau)\varphi(\tau)) - 2s + \log h}\Big)$$

$$\cancel{p^*\rho_\Sigma - \frac{[(1+\tau)\varphi]'}{2(1+\tau)}p^*\omega_\Sigma}$$

$p^*\omega_\Sigma$?

$$= p^*\rho_\Sigma - i\partial\bar\partial\left(-2s+\log h + \log((1+\tau)\varphi(\tau))\right)$$

$$= p^*\rho_\Sigma + 2i\partial\Big(\frac{1}{2\bar w}$$

$$-i\partial\left(\frac{\varphi}{2\bar w}\frac{\partial}{\partial\tau}\left(\log((1+\tau)\varphi(\tau))\right)d\bar w\right) = -i\partial\left(\frac{\varphi}{2\bar w}\cdot\frac{[(1+\tau)\varphi]'}{1+\tau}\,d\bar w\right)$$

$$= -i\left(\frac{1}{4|w|^2}\varphi\frac{\partial}{\partial\tau}\left[\frac{[(1+\tau)\varphi]'}{1+\tau}\right]dw\wedge d\bar w\right)$$

$$= \frac{-i\varphi}{4|w|^2}\left[\frac{(1+\tau)[(1+\tau)\varphi]'' - [(1+\tau)\varphi]'}{(1+\tau)^2}\right]dw\wedge d\bar w$$

$$\rho = p^*\rho_\Sigma - \frac{[(1+\tau)\varphi]'}{2(1+\tau)}p^*\omega_\Sigma - \frac{\varphi}{2}\left[\quad\right]\frac{i\,dw\wedge d\bar w}{|w|^2}$$

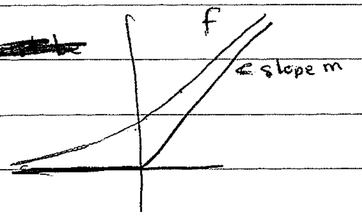
$$\Rightarrow \quad S(\omega) = \mathrm{Tr}_\omega\rho_\Sigma = \frac{-2}{1+\tau} - \frac{1}{2(1+\tau)}[(1+\tau)\varphi]''$$

- Fourth order non-linear PDE $\longrightarrow$ 2nd order linear ODE for momentum profile
- Seek to extremize Calabi ~~metric~~ functional over metrics of this type ~~in this class~~
- Later show that these are extremizers over whole Kähler class

~~to bdry~~ What conditions must be imposed on φ?

Fact: under ω/g, fibers are totally geodesic (just compute $\nabla_w \frac{\partial}{\partial w}$)

~~[illegible]~~



$\Rightarrow \quad \tau \in [0, m]$, with $\tau = 0$ as $s \to -\infty$ and $\tau = m$ as $s \to \infty$

Want g to be smooth on ruled surface / complete on ruled surface $\smallsetminus$ inf. of zero section

$g_{\text{fiber}} = \varphi(\tau) \left| \frac{dw}{w} \right|^2 = \left(\frac{\varphi(\tau)}{r} \right) h \, |dw|^2$ where $r = |w|^2 h(z) = e^{2s} \to 0$ as $s \to -\infty$, or $\tau \to 0$

$\frac{\varphi(\tau)}{r}$ must have finite, pos. limit as $\tau \to 0$, say (sim for $\tau \to m$)

$s \overset{+C}{=} \int_{\tau_0}^{\tau} \frac{d\tau}{ds} \, ds = \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}$, Suppose $\varphi(x) = a_1 x + O(x^2)$

$= \int_{\tau_0}^{\tau} \frac{dx}{a_1 x + O(x^2)}$

$= \int_{\tau_0}^{\tau} \frac{1}{a_1} \frac{1}{x} + b_0 + b_1 x + \dots \, dx$

$s = \frac{1}{2} \log r \qquad e^{\frac{1}{4} \log^2 r}$

$r \quad \varphi(\tau) = \frac{1}{F''(\tau)}, \quad F''(\tau) = \frac{\partial s}{\partial \tau}, \quad F'(\tau) = s, \quad s \overset{+C}{=} \int^{\tau} \frac{dx}{\varphi(x)} = \frac{1}{a_1} \log \tau + O(\tau^{-1}) \quad \Rightarrow \log \tau = a_1 s + C + s^2 ?$

$\Rightarrow \tau = a r^{a_1/2}$

$= \frac{1}{a_1} \log \tau + b_0 \tau + \dots$

$\Rightarrow$ ~~[illegible]~~ $e^{2s} = a e^{2/a_1 \log \tau} e^{O(\tau)} \to a \tau^{2/a_1}$ as $\tau \to 0$

$\Rightarrow g_{\text{fiber}}$ finite and pos. iff $a_1 = 2$

otherwise have ~~blow up~~ sing. at 0 section, w/ finite volume complete metric for φ vanishing to order 2

Sim. reasoning for $\varphi'(m) = -2$ or $\varphi'(m) = 0$

$\varphi + \varepsilon\tilde{\varphi} \in A$ where $\varphi > 0$ can take $\tilde{\varphi}$ pos or neg $\Rightarrow S'' \leq 0$

S cts since S concave $\Rightarrow \varphi \in C^2$

Minimizers unique

$A = \{ \varphi \in \cancel{H^2} \ \ S[0,m]\to\mathbb{R} \mid \varphi \in H^2,\ \varphi \geq 0$, satisfies bdry cond. $\}$

$D\,Cal_\varphi(\tilde{\varphi}) = -\int (S(\varphi) - S(\Phi))[(1+\tau)\tilde{\varphi}]'' d\tau$, $\quad \tilde{\varphi} = \tilde{\varphi}' = 0$ at bdry, $\tilde{\varphi} \geq 0$

$= -\int_0^m S(\varphi)'' \tilde{\varphi}(1+\tau) d\tau \geq 0 \quad \Rightarrow S''$ a neg. dist.

$Cal(\psi) \leq Cal(\psi) + \int (S(\varphi) - S(\psi))^2 (1+\tau) d\tau$

$= Cal(\psi) + 2\int (S(\varphi) - S(\psi)) S(\psi)(1+\tau) d\tau$

$= Cal(\psi) + \int [(1+\tau)\varphi - (1+\tau)\psi]'' S(\psi)\, d\tau$

$= Cal(\psi) + \int \varphi\, S(\psi)''(1+\tau) d\tau$

$\leq Cal(\varphi)$

$\cancel{Cal(\varphi)\ \text{min}} \Rightarrow \int (S(\varphi) - S(\psi))^2 (1+\tau) d\tau = 0 \Rightarrow S(\varphi) = S(\psi) \Rightarrow \varphi = \psi.$

<u>Equation for extremal metric:</u>

Seek to minimize functional

~~Calabi energy on X proportional to~~ $Cal(\varphi) = \int_0^m (S(\varphi) - S(\Phi))^2 (1+\tau) d\tau$ (bounds H^2 norm of φ) (in $\mathbb{R}^1$, H^2 norm bds C^1 $\Rightarrow$ bdry cond. preserved)

differs from L^2 norm of $S(\varphi)$ (wrt meas. $(1+\tau)d\tau$) by a const.

where Φ is a momentum profile (satisfying bdry cond.)

st $S(\Phi)'' = 0$

This is the condition of an extremizer

since extremal metrics have $\partial_{\bar{k}}(g^{i\bar{j}} \partial_j S) = 0 \iff \varphi S(\varphi)'' = 0$ in our coords.

NEED CONVERSE: $\psi S(\psi)'' = 0$, $S(\psi)$ concave $\Rightarrow \psi = \varphi$, minimizer

<u>Explicit minimizers:</u>

<u>Case 1:</u> $m < k_1 \approx 18.889$

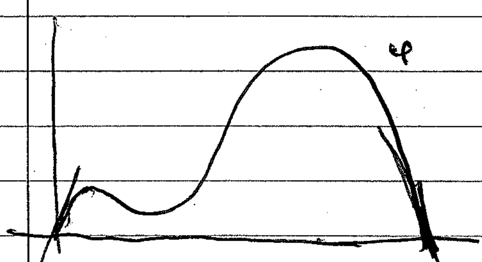
$$\frac{1}{2(1+\tau)}\left(-4 - [(1+\tau)\varphi]''\right) = A\tau + B \quad \Rightarrow \quad (1+\tau)\varphi = -\frac{A\tau^4}{6} - \frac{(A+B)\tau^3}{3} - B\tau^2 - 2\tau^2 + C\tau + D$$

bdry cond. give lin. eq. on A, B, C, D

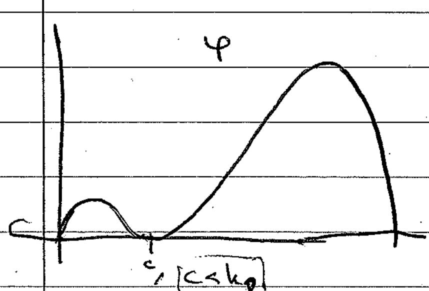
get $\varphi(\tau) = \dfrac{2\tau(m-\tau)}{m(m^2+6m+6)(1+\tau)}\left[\tau^2(2m+2) + \tau(-m^2+4m+6) + m^2+6m+6\right]$

For φ to be pos. need exp. in brackets pos,

true iff $m < k_1$, k_1 the pos. real root of $m^4 - 16m^3 - 52m^2 - 48m - 12$



<u>Case 2:</u> $k_1 \leq m \leq k_2(k_2+2) \approx 35.33$

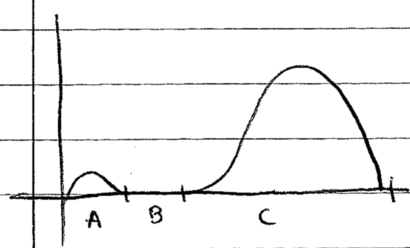


Solve for φ in smaller intervals w/ $\varphi' = 0$ bdry cond.

$\Rightarrow$ mfld breaks up into pieces

(Note: S is singular at $\tau = 0$ unless $\varphi'(0) = 2$)

Case 3: $m > k_2(k_2+2)$



X breaks up into 3 pieces

on B, S^1 orbits collapse

These are minimizers over entire Kähler class:

Must use test config inequality:

(Donaldson): χ a test config ~~for~~ (X, L) st $F(\chi)$ neg. Then for any metric

Futaki inv.

ω in $c_1(L)$ we have

$$\|S(\omega) - \bar{S}\|_{L^2} \geq \frac{-F(\chi_\xi)}{\|\chi_\xi\|}$$

Our $\omega \in c_1(L_m = C + mS_\infty)$ (makes sense for m rational)

Can find seq. of test configs χ_i st $\lim_{i\to\infty} \frac{-F(\chi_i)}{\|\chi_\xi\|} = \|S(\varphi) - \bar{S}\|_{L^2}$ (see Adam's talk?)

Calabi Flow: The flow ~~starting~~ from a metric given by the momentum ~~in the Calabi ansatz~~ construction remains ~~in~~ that form, exists for all time, and converges to the extremal metric exponentially fast

$$\omega_t = p^* \omega_\Sigma + 2i\partial\bar{\partial} f_t(s) \Rightarrow \text{Calabi flow: } \frac{\partial f_t(s)}{\partial t} = S(\omega_t)$$

$F_t^{(\tau)} = \hat{f}_t(\tau)$: $\quad f(s) + F(\tau) = s\tau \Rightarrow \frac{\partial F_t}{\partial t} = -\frac{\partial f_t}{\partial t} = -S$, $\quad \cancel{\varphi = \frac{1}{F_t''(\tau)}, \frac{\partial F''}{\partial t} = \frac{\partial^2}{\partial \tau^2}\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial \tau}\left(\varphi \frac{\partial f}{\partial s}\right) = -\varphi \frac{\partial \dot{f}}{\partial s} - \varphi^3 \dot{\varphi}}$

$$\Rightarrow \boxed{\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t}\frac{1}{F''(\tau)} = -\frac{1}{(F'')^2}\dot{F}'' = -\varphi_t^2 \cdot (-S'') = \varphi_t^2 S(\varphi_t)''}$$

$$\frac{d}{dt}\left(\frac{-1}{2(1+\tau)}\left[(1+\tau)\varphi\right]''\right) = \frac{-1}{2(1+\tau)}\left[(1+\tau)\varphi^2 S''\right]''$$

Mabuchi functional $\mathcal{M} = \int_0^1 dt \int \dot{\varphi}_t (R-\bar{R}) \det g_{\varphi_t} = \int dt \int (R-\bar{R})^2$

<u>Lemma:</u> φ a solution to the Calabi flow

$$\Rightarrow \frac{d\,\mathrm{Cal}(\varphi)}{dt} = \frac{d}{dt}\int_0^m (S(\varphi_t)-S(\Phi))^2(1+\tau)\,d\tau$$

$$= 2\int_0^m (S(\varphi)-S(\Phi))\left(-\frac{1}{2(1+\tau)}\left[(1+\tau)\varphi^2 S(\varphi)''\right]''\right)(1+\tau)\,d\tau$$

Int. by parts $2x$, $S(\Phi)''=0$

$$= -\int_0^m \varphi^2 (S(\varphi)'')^2(1+\tau)\,d\tau \qquad \text{since } \varphi^2 \ \&\ (\varphi^2)' = 0 \text{ at } 0 \,\&\, m$$

$\Rightarrow H^2$ norm of φ decreasing

<u>Prop:</u> If the flow exists for all time, then the momentum profiles converge in H^2 to the minimiser found above.

<u>Pf:</u> Write Ψ for the minimizer (the momentum profile of an extremal metric only in case 1)

Let $\mathcal{M}(\varphi) = \int_0^m \left(\frac{\Psi}{\varphi} + \log\varphi\right)(1+\tau)\,d\tau$ on momentum profiles φ

(modified Mabuchi functional in case 1)

$$\frac{d\mathcal{M}}{dt} = \int_0^m (-\Psi S'' + \varphi S'')(1+\tau)\,d\tau$$

$$= \int_0^m (\varphi-\Psi)(S(\varphi)-S(\Psi))''(1+\tau)\,d\tau + \int_0^m \varphi S(\Psi)''(1+\tau)\,d\tau \qquad \text{since } \Psi S(\Psi)''=0$$

$S(\Psi)''$ a neg. distribution

$$\leq -2\int_0^m (S(\varphi)-S(\Psi))^2(1+\tau)\,d\tau$$

But also $\mathcal{M}(\varphi) \geq \int_0^m \log\varphi\cdot(1+\tau)\,d\tau \geq -C_1\int_0^m \log\frac{\Theta}{\varphi}\,d\tau - C_2$ where Θ is a fixed momentum profile

$$\geq -C_3 \log\int_0^m \frac{\Theta}{\varphi}\,d\tau - C_4 \qquad \text{since log concave}$$

Will show that then $\mathcal{M}(\varphi_t) \geq -C\log(1+t) - D$

then $\mathcal{M}$ is decreasing and greater than a fn whose deriv. goes to 0

Hence, along a subseq. $\mathcal{M}'(\varphi_{t_n})$ goes to 0 $\Rightarrow$ $S(\varphi_t) \to S(\Psi)$ in L^2, w $\|S(\varphi_t)\|_{L^2}$

$\Rightarrow \lim \|S(\varphi_t)\| = \|S(\Psi)\|_{L^2}$

$\|\varphi\|_{H^2} \leq C\|S\|_{L^2} \Rightarrow \exists$ some subseq. φ_{t_i} converging strongly ~~weakly~~ in H^2 ~~to some limit~~ w/ minimum $\|S\|_{L^2}$

The minimizer is unique $\Rightarrow$ the limit is Ψ

Lemma: $\Theta: [0,m] \to \mathbb{R}$ a momentum profile. For a solution φ_t of the Calabi flow we have

$$\int_0^m \frac{\Theta}{\varphi_t}\, d\tau < C(1+\tau) \quad \text{for some const } C.$$

Pf: Let $\mathcal{F}(\psi) = \int_0^m \left(\frac{\Theta}{\psi} - \log\frac{\Theta}{\psi}\right) d\tau$ for any momentum profile

Along the flow, $\frac{d}{dt}\mathcal{F}(\varphi_t) = \int_0^m -\Theta S(\varphi_t)'' + \varphi S(\varphi_t)''\, dt = \int_0^m (\varphi_t - \Theta)'' S(\varphi_t)\, d\tau$

$$\leq \left(\int_0^m (\varphi_t'' - \Theta'')^2 d\tau\right)^{1/2} \left(Cal(\varphi_t) + C\right)^{1/2}$$

$$\leq C$$

φ_t has uniform H^2 bd, $Cal(\varphi_t)$ bd'd $\Rightarrow$ $\mathcal{F}(\varphi_t) \leq C(1+t)$ for some $C > 0$

$x - \log x > x/2 \Rightarrow \int \frac{\Theta}{\varphi_t} d\tau \leq C(1+\tau)$

Note: $\mathcal{M}$ bd'd below except in Case 3 (X breaks up into 3 parts) $m \geq k_2(k_2+2)$

since

$$\mathcal{M}(\varphi) = \int\left(\frac{\Psi}{\varphi} - \log\frac{\Psi}{\varphi}\right)(1+\tau)\,d\tau + \int \log\Psi\,(1+\tau)\,d\tau$$

If Ψ vanishes at finitely many pts to finite order, then $\int \log\Psi$ finite

~~$\log x < x$~~ $x > \log x$

$$\Rightarrow \mathcal{M}(\varphi) \geq \int_0^m \log\Psi\,(1+\tau)\,d\tau$$

Note: Must show long term existence; not clear apriori that flow preserves positivity of φ